pressure changes of infinitesimal amplitude are impressed, will be taken into account explicitly by means of the formal theory of finite strain, to justify Eq. (1) for a Debye solid and Eq. (2) for a Druyvesteyn-Meyering solid. Thus, any restriction in the preceding discussion to the case of infinitesimal strain will be lifted.

A. Debye Solid

For finite deformation, under hydrostatic pressure alone, of an isotropic elastic solid about the arbitrary point (V_1,P_1) on its pressure-volume curve, Murnaghan has shown[17] that the change $P-P_1$ in pressure of the silid from the point (V_1,P_1) to the point (V,P) is given by a Taylor series through second-order terms in a parameter e as

$$P-P_1=(3\lambda+2\mu+P_1)e-\tfrac{1}{2}(18l+2n-6\lambda-4\mu-3P_1)e^2, \quad (37)$$

where λ and μ are Lamé parameters evaluated at the point (V_1,P_1), and l and n are Murnaghan parameters corresponding to the same point. The variable e is connected with the volumes by the exact relation

$$1-2e=(V/V_1)^{2/3}, \quad (38)$$

which yields

$$e=-\frac{1}{3}\frac{V-V_1}{V_1}+\frac{1}{18}\left(\frac{V-V_1}{V_1}\right)^2-\frac{2}{81}\left(\frac{V-V_1}{V_1}\right)^3 \quad (39)$$

by a power-series expansion.

From the definition (10) of the bulk modulus K, Eq. (37) yields

$$K=K_1-3(V\partial K/\partial V)_1 e, \quad (40)$$

where

$$K_1=\lambda+\tfrac{2}{3}\mu+\tfrac{1}{3}P_1, \quad (41a)$$

$$(V\partial K/\partial V)_1=2l+(2/9)n-(1/9)P_1. \quad (41b)$$

One notes that inclusion of the second-order term in Eq. (37) for P makes the graph of $P-P_1$ against the dilatation $(V-V_1)/V_1$ a parabola, instead of the straight line corresponding to the first-order term in e. The presence of the finite pressure introduces the correction term P_1 to $3\lambda+2\mu$ in the first term of Eq. (37) for $P-P_1$, which, by Eq. (41a), changes the physical interpretation of the Lamé parameters in terms of the bulk modulus at finite pressure, as compared to the interpretation of Eq. (11) for infinitesimal pressure. It must be emphasized that the Lamé parameters λ and μ, and the Murnaghan parameters l and n, are functions of P_1, in general.

By a fundamental theorem of Murnaghan,[17] an elastic body which is initially isotropic remains so when subjected to a finite strain due to hydrostatic pressure alone; the initial state (V_1,P_1) above must be produced in this manner. If a general infinitesimal stress is superposed in this situation, the body remains approximately isotropic. Hughes and Kelly[36] have extended a prior result of Murnaghan[17] to show that the response of the solid to the superposed infinitesimal stress in this case is completely specified by two generalized Lamé parameters L and M, in a manner entirely analogous to the specification by λ and μ in the infinitesimal case. The values of L and M are given by

$$L=\lambda+P_1-(6l-2m+n-2\lambda-2\mu-P_1)e, \quad (42a)$$

$$M=\mu-P_1-(3m-\tfrac{1}{2}n+3\lambda+3\mu+P_1)e, \quad (42b)$$

in which m, like l and n, is a Murnaghan parameter evaluated at (V_1,P_1).

The speeds C_l and C_t of longitudinal and transverse waves, respectively, of infinitesimal amplitude superposed on a state of finite strain due to hydrostatic pressure, are given by equations analogous to Eqs. (9) in the infinitesimal case, as

$$C_l^2=(L+2M)/\rho, \quad C_t^2=M/\rho, \quad (43)$$

where ρ is the density corresponding to the volume V. Hughes and Kelly give expressions for L and M which omit terms in P_1, since these authors referred the body to an initial state of zero pressure, for experimental purposes. If use is made of the relation $\rho=\rho_0(1+3e)$ obtained from Eq. (39), for ρ in terms of an initial density ρ_0, Eqs. (43) reduce to the corresponding expressions of Hughes and Kelly for $P_1=0$, and agree with the corresponding relations of Brillouin.

With K given by Eq. (40), the values of L and M satisfy the relation

$$K=L+\tfrac{2}{3}M, \quad (44)$$

analogous to Eq. (11) in the infinitesimal case. The expression (12) for the Poisson ratio in the infinitesimal case must be replaced for finite strain by a generalized Poisson ratio Σ defined by

$$\Sigma=\tfrac{1}{2}L/(L+M). \quad (45)$$

The stability conditions[24] $K,\ M\geq 0$ require that $\Sigma\leq\frac{1}{2}$, and one obtains $\Sigma\rightarrow\sigma$ in the limit $P_1,\ P\rightarrow 0$. With introduction of Σ, the response of the solid under finite strain to a superposed infinitesimal stress of general type can be described completely by the two parameters K and Σ, instead of L and M.

Use of Eqs. (44) and (45) in the analog of Eq. (8) obtained by replacing c_l and c_t by C_l and C_t, respectively, yields

$$\nu_D=SN^{1/3}M^{-1/2}K^{1/2}V^{1/6} \quad (46)$$

for the Debye frequency ν_D, where $S=s_D(\Sigma)$ in terms of s_D of Eq. (14). Corresponding to the case of Sec. IIA, it is necessary that Σ be constant to satisfy the Grüneisen postulate that the frequencies of the longitudinal and transverse waves show the same volume variation. Under this assumption, the definition (4)

[36] D. S. Hughes and J. L. Kelly, Phys. Rev. 92, 1145 (1953).